Extended rank system

Applicable for any types of team games

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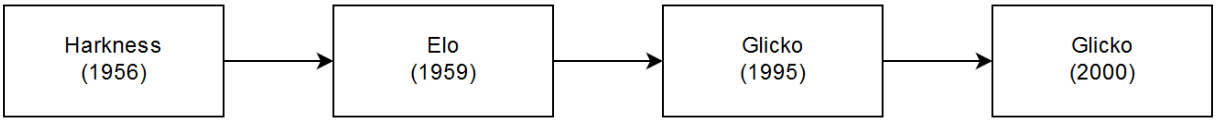
1. Introduction

Rating systems along the history were different, but all of them were trying to compare two unknown people in one field. Probably the most famous attempt to that problem was Elo rating system[1], widely used in chess games. But Elo rating system was improved and introduced as Glicko rating system[2]. The full improvements history can be shown as[3], [4]:



2. Power of a player

It is known that two players in the game can differ a lot with their power in the competition. For them it can be even several orders of magnitude of power difference (like 1 over 1000). But the pointing systems above are handling with that. As we can read in Wikipedia[1]:

$$Q = 10^{\frac{R}{400}}$$

Where Q - power of a player, R - his/her rating points

That means if there are two players with a difference of 400 points, the stronger one should be exactly 10 times stronger than the weaker one.
And since 10^(1/400) ≈ 1.005773063, the equation can be simplified to:

$$Q = 1.005773063^R$$
$$\beta = 1.005773063$$
$$Q = \beta^R$$

3. Two player teams

Since Widelands is not simple 1 vs 1 game only, we need to calculate the correct average of the players points. And since the power of the players is not linear, but exponential, the average power of two players is not a simple arithmetic mean.

To get proper equation, it is needed to determine the goal. For two players with the points R_1 and R_2, it is needed to find two exactly equal opponents with rating R_3 that their total power is equal to total power of R_1 and R_2:

$$Q_1 + Q_2 = Q_3 + Q_3 = 2 \cdot Q_3$$

Using equation above we get:

$$Q_1 + Q_2 = \beta^{R_1} + \beta^{R_2}$$
$$2 \cdot Q_3 = 2 \cdot \beta^{R_3}$$
$$\beta^{R_1} + \beta^{R_2} = 2 \cdot \beta^{R_3}$$
$$\frac{1}{2}(\beta^{R_1} + \beta^{R_2}) = \beta^{R_3}$$

Solving the equation is by using natural logarithm[5] ln on both sides:

$$\ln\left(\frac{1}{2}(\beta^{R_1}+\beta^{R_2})\right)=\ln(\beta^{R_3})$$
$$\ln(\beta^{R_1}+\beta^{R_2})-\ln(2)=R_3\cdot\ln(\beta)$$
$$R_3=\frac{\ln(\beta^{R_1}+\beta^{R_2})-\ln(2)}{\ln(\beta)}$$

It can be also written as (assuming $R_1 < R_2$):

$$R_3=R_1+\frac{\ln(1+\beta^{R_2-R_1})}{\ln(\beta)}-\frac{\ln(2)}{\ln(\beta)}$$

4. Examples of usage

4.1. Same power

Let's assume that $R_1 = R_2$, then:

$$R_3=\frac{\ln(\beta^{R_1}+\beta^{R_2})-\ln(2)}{\ln(\beta)}=\frac{\ln(\beta^{R_1}+\beta^{R_1})-\ln(2)}{\ln(\beta)}=\frac{\ln(2\cdot\beta^{R_1})-\ln(2)}{\ln(\beta)}=$$
$$=\frac{\ln(2)+\ln(\beta^{R_1})-\ln(2)}{\ln(\beta)}=\frac{\ln(\beta^{R_1})}{\ln(\beta)}=\log_\beta(\beta^{R_1})=R_1$$

So as we expected, the formula gets us to the same rating points as the initial players.

4.2. Difference of 200 (decent difference, but not huge one)

Let's assume that $R_2 = R_1+200$, then:

$$R_3=R_1+\frac{\ln(1+\beta^{R_2-R_1})}{\ln(\beta)}-\frac{\ln(2)}{\ln(\beta)}=R_1+\frac{\ln(1+\beta^{R_1+200-R_1})}{\ln(\beta)}-\frac{\ln(2)}{\ln(\beta)}=$$
$$=R_1+\frac{\ln(1+\beta^{200})}{\ln(\beta)}-\frac{\ln(2)}{\ln(\beta)}=R_1+\frac{\ln(1+3.162278\ldots)}{\ln(1.00577\ldots)}-\frac{\ln(2)}{\ln(1.00577\ldots)}=$$
$$=R_1+247.732\ldots-120.412\ldots\approx R_1+127.3$$

So the result is that the expected power of two players is about 127 points higher than the weaker player or about 73 points lower than the stronger one.

4.3. Finding difference that doesn't affect the stronger player

Last part is about finding the rating that is not affecting the result rating of stronger player and the weaker one. In mathematical language can be written as:

$$R_1+R_2\approx 0+R_2$$

To finish this equation we need to define approximately sign. From a rating system, two players are equal power when their amount of points differs 1 at most. Applying that:

$$2 \cdot R_3 = R_2 + 0$$
$$2 \cdot (R_3 + 1) = R_2 + R_1$$

So let's find out what R_3 equals to:

$$R_3 = \frac{\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2)}{\ln(\beta)} = \frac{\ln(\beta^{0} + \beta^{R_2}) - \ln(2)}{\ln(\beta)} = \frac{\ln(1 + \beta^{R_2}) - \ln(2)}{\ln(\beta)}$$

Since R_2 is expected to be around 1000 or more, the value of β^{R_2} is expected to be much greater than 1:

$$\beta^{R_2} \gg 1$$

So applying that, it can be calculated:

$$R_3 = \frac{\ln(1 + \beta^{R_2}) - \ln(2)}{\ln(\beta)} \approx \frac{\ln(\beta^{R_2}) - \ln(2)}{\ln(\beta)} = R_2 - 120.412 \ldots$$

The result is that half of power of sufficient rating score is equal to score decreased by about 120 points. Going further, we apply those numbers to the second equation:

$$R_3 + 1 = R_2 - 120.412 + 1 = R_2 - 119.412 = R_2 - \gamma$$
$$\gamma = 119.412 \ldots$$
$$R_2 + R_1 = \frac{\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2)}{\ln(\beta)} = R_2 - \gamma$$
$$\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2) = R_2 \cdot \ln(\beta) - \gamma \cdot \ln(\beta)$$
$$\ln(\beta^{R_1} + \beta^{R_2}) = R_2 \cdot \ln(\beta) - \gamma \cdot \ln(\beta) + \ln(2)$$
$$\beta^{R_1} + \beta^{R_2} = \exp(R_2 \cdot \ln(\beta) - \gamma \cdot \ln(\beta) + \ln(2))$$
$$\beta^{R_1} = \exp(R_2 \cdot \ln(\beta) - \gamma \cdot \ln(\beta) + \ln(2)) - \beta^{R_2}$$
$$R_1 = \frac{\ln(\exp(R_2 \cdot \ln(\beta) - \gamma \cdot \ln(\beta) + \ln(2)) - \beta^{R_2})}{\ln(\beta)}$$
$$= \frac{\ln\left(\frac{\exp(R_2 \cdot \ln(\beta))}{\exp(\gamma \cdot ln(\beta))} \cdot \exp(\ln(2)) - \beta^{R_2}\right)}{\ln(\beta)} =$$
$$= \frac{\ln\left(\frac{\beta^{R_2}}{\beta^{\gamma}} \cdot 2 - \beta^{R_2}\right)}{\ln(\beta)} = \frac{\ln\left(\beta^{R_2}\left(\frac{2}{\beta^{\gamma}} - 1\right)\right)}{\ln(\beta)} = \frac{\ln(\beta^{R_2}) + \ln\left(\frac{2}{\beta^{\gamma}} - 1\right)}{\ln(\beta)} =$$
$$= \frac{\ln(\beta^{R_2})}{\ln(\beta)} + \frac{\ln\left(\frac{2}{\beta^{\gamma}} - 1\right)}{\ln(\beta)} = \frac{\mathrm{R}_2 \cdot \ln(\beta)}{\ln(\beta)} + \frac{\ln\left(\frac{2}{\beta^{\gamma}} - 1\right)}{\ln(\beta)} = R_2 + \frac{\ln\left(\frac{2}{\beta^{\gamma}} - 1\right)}{\ln(\beta)}$$
$$R_1 \approx R_3 - 895.44$$

So if the weaker player has lower rating by about 900 points, this doesn't matter to global rating for both players.

4.4. Comparing two teams with ranks differed by exactly the same amount of points

Let's assume that there are two teams: R_1 and R_2 versus R_3 and R_4. And $R_3 = R_1 + d$, $R_4 = R_2 + d$. Expected result is that average R_1 and R_2 is exactly lower than average R_3 and R_4 by *d* points.

$$R_{12} = R_1 + R_2 = \frac{\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2)}{\ln(\beta)}$$
$$R_{34} = R_3 + R_4 = \frac{\ln(\beta^{R_3} + \beta^{R_4}) - \ln(2)}{\ln(\beta)} = \frac{\ln\left(\beta^{R_1+d} + \beta^{R_2+d}\right) - \ln(2)}{\ln(\beta)} =$$
$$= \frac{\ln\left(\beta^{R_1} \cdot \beta^d + \beta^{R_2} \cdot \beta^d\right) - \ln(2)}{\ln(\beta)} = \frac{\ln\left(\beta^d(\beta^{R_1} + \beta^{R_2})\right) - \ln(2)}{\ln(\beta)} =$$
$$= \frac{\ln\left(\beta^d\right) + \ln(\beta^{R_1} + \beta^{R_2}) - \ln(2)}{\ln(\beta)} = \frac{\ln\left(\beta^d\right)}{\ln(\beta)} + \frac{\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2)}{\ln(\beta)} =$$
$$= \frac{\ln\left(\beta^d\right)}{\ln(\beta)} + R_{12} = d + R_{12}$$

So increasing rank of every player in a team by *d* points, it will increase average rank by *d* points.

5. Calculating Rating Deviation

Glicko and Glicko-2 rating systems contain two variables that define the rank position and player's strength: R - rating points and RD - rating deviation. The second one is a standard deviation of the first value.

To calculate standard deviation of a function of two variables we can use an equation[6]:

$$\sigma_t = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \cdot \sigma_1^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \cdot \sigma_2^2}$$

And in our case this formula can be written as:

$$RD_3 = \sqrt{\left(\frac{\partial R_3}{\partial R_1}\right)^2 \cdot RD_1^2 + \left(\frac{\partial R_3}{\partial R_2}\right)^2 \cdot RD_2^2}$$

The only problem is about calculating the derivatives:

$$\frac{\partial R_3}{\partial R_i} = \frac{\partial}{\partial R_i}\left(\frac{\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2)}{\ln(\beta)}\right)$$
$$\frac{\partial R_3}{\partial R_i} = \frac{\beta^{R_i}}{\beta^{R_1} + \beta^{R_2}}$$
$$RD_3 = \sqrt{\left(\frac{\beta^{R_1}}{\beta^{R_1} + \beta^{R_2}}\right)^2 \cdot RD_1^2 + \left(\frac{\beta^{R_2}}{\beta^{R_1} + \beta^{R_2}}\right)^2 \cdot RD_2^2}$$

6. Example

Let's see what happened when we have: R_1 = 1700, RD_1 = 50, R_2 = 1300, RD_2 = 150.

$$R_3 = \frac{\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2)}{\ln(\beta)}$$
$$R_3 = \frac{\ln(\beta^{1700} + \beta^{1300}) - \ln(2)}{\ln(\beta)}$$
$$R_3 \approx 1596$$

$$RD_3 = \sqrt{\left(\frac{\beta^{R_1}}{\beta^{R_1} + \beta^{R_2}}\right)^2 \cdot RD_1^2 + \left(\frac{\beta^{R_2}}{\beta^{R_1} + \beta^{R_2}}\right)^2 \cdot RD_2^2}$$
$$RD_3 = \sqrt{\left(\frac{\beta^{1700}}{\beta^{1700} + \beta^{1300}}\right)^2 \cdot 50^2 + \left(\frac{\beta^{1300}}{\beta^{1700} + \beta^{1300}}\right)^2 \cdot 150^2}$$
$$RD_3 \approx 47.5$$

So the average rating of two players like this is 1596 and their rating deviation is 47.5.

7. Generalization: more players in a team

Widelands contains games up to 8 players, so it can be up to 4 players in two average size teams. Also it can be played as 1 vs 7, so the largest possible team can contain up to 7 players. To calculate average ratings for such games we need general formula.

Sticking to point 3. way of thinking, we can calculate the average like this:

$$Q_1 + Q_2 + Q_3 + \ldots + Q_n = \sum_{i=1}^{n} Q_i = n \cdot Q_t$$
$$Q_i = \beta^{R_i}$$
$$\sum_{i=1}^{n} \beta^{R_i} = n \cdot \beta^{R_t}$$
$$R_t = \frac{\ln(\sum_{i=1}^{n} \beta^{R_i}) - \ln(n)}{\ln(\beta)}$$

$$DR_t = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial R_t}{\partial R_i}\right)^2 \cdot RD_i^2}$$
$$\frac{\partial R_t}{\partial R_i} = \frac{\beta^{R_i}}{\sum_{j=1}^{n} \beta^{R_j}}$$
$$DR_t = \sqrt{\sum_{i=1}^{n} \left(\frac{\beta^{R_i}}{\sum_{j=1}^{n} \beta^{R_j}}\right)^2 \cdot RD_i^2}$$

So in conclusion of this point, if there are two teams with the same number of players, the easiest way of thinking about their ranks it to calculate the average rank and rank deviation from formulas above. Then, after the game is won or lost, the expected gain or lose of rank points should be added to all of the players' ranks.

8. Null player

To introduce next steps we need to consider player in teams of two where one of the players has significantly larger rank than the lower one. According to point 4.4. the difference should be over 900 to fulfill this requirement. Such a player will be called "null player", as it doesn't affect positively average power and such a player doesn't help the team in the game.

8.1. Average rank points

Calculating average rank points in this situation we will have:

$$R_3 = \frac{\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2)}{\ln(\beta)}$$
$$\beta^{R_1} \gg \beta^{R_2}$$
$$R_3 \approx \frac{\ln(\beta^{R_1}) - \ln(2)}{\ln(\beta)} = \frac{\ln(\beta^{R_1})}{\ln(\beta)} - \frac{\ln(2)}{\ln(\beta)} = \frac{R_1 \cdot \ln(\beta)}{\ln(\beta)} - \frac{\ln(2)}{\ln(\beta)} = R_1 - \frac{\ln(2)}{\ln(\beta)}$$

This equation is the more accurate, the greater difference between R_1 and R_2 is. And if we want to simulate a player who has no rank at all, this equation shows how the rank will behave with such players. As it can be seen, average rank can be calculated as rank of a stronger player, lowered by a constant value $\ln(2)/\ln(\beta) \approx 120.4$.

To see what changed, let's see how the equation is different when we want to calculate average for only one player:

$$R_1 = \frac{\ln(\beta^{R_1}) - \ln(1)}{\ln(\beta)} = R_1 - \frac{\ln(1)}{\ln(\beta)}$$

And since ln(1)=0, the subtraction cancels out. But to comparison purposes it is shown here. The main difference is that ln(1) changed into ln(2).

8.2. Rating deviation

To move forward analysis on this topic, we can calculate how rating deviation DR will change when null player will attend a person and create a team:

$$RD_t = \sqrt{\left(\frac{\beta^{R_1}}{\beta^{R_1} + \beta^{R_n}}\right)^2 \cdot RD_1^2 + \left(\frac{\beta^{R_n}}{\beta^{R_1} + \beta^{R_n}}\right)^2 \cdot RD_n^2}$$

$$\beta^{R_n} \ll \beta^{R_1}$$

$$RD_t \approx \sqrt{\left(\frac{\beta^{R_1}}{\beta^{R_1}}\right)^2 \cdot RD_1^2 + \left(\frac{\beta^{R_n}}{\beta^{R_1}}\right)^2 \cdot RD_n^2} \approx \sqrt{(1)^2 \cdot RD_1^2 + (0)^2 \cdot RD_n^2} =$$

$$= \sqrt{RD_1^2} = RD_1$$

The result tells us that the rating deviation isn't affected by a null player's deviation.

8.3. Multiple null players at once

Last thing is about adding more null players to a team of some normal players. For this calculation it is assumed that there are *n* normal players and *d* null players. Then the rating average will be as follows:

$$R_t = \frac{\ln\left(\sum_{i=1}^{n} \beta^{R_i} + \sum_{i=1}^{d} \beta^{R_n}\right) - \ln(n+d)}{\ln(\beta)} = \frac{\ln(\sum_{i=1}^{n} \beta^{R_i} + d \cdot \beta^{R_n}) - \ln(n+d)}{\ln(\beta)}$$

$$\beta^{R_n} \ll \beta^{R_i}$$

$$R_t \approx \frac{\ln(\sum_{i=1}^{n} \beta^{R_i}) - \ln(n+d)}{\ln(\beta)}$$

To conclude this point, it is needed to compare it with an original equation for R_t (without null players):

$$R_{t0} = \frac{\ln(\sum_{i=1}^{n} \beta^{R_i}) - \ln(n)}{\ln(\beta)}$$

And to join the equation, simple change can be applied:

$$R_t = \frac{\ln(\sum_{i=1}^{n} \beta^{R_i}) - \ln(n+d)}{\ln(\beta)} = \frac{\ln(\sum_{i=1}^{n} \beta^{R_i}) - \ln(n) + \ln(n) - \ln(n+d)}{\ln(\beta)} =$$

$$= \frac{\ln(\sum_{i=1}^{n} \beta^{R_i}) - \ln(n)}{\ln(\beta)} + \frac{\ln(n) - \ln(n+d)}{\ln(\beta)} = R_{t0} - \frac{\ln(n+d) - \ln(n)}{\ln(\beta)} =$$

$$= R_{t0} - \frac{\ln\left(\frac{n+d}{n}\right)}{\ln(\beta)} = R_{t0} - \frac{\ln\left(1 + \frac{d}{n}\right)}{\ln(\beta)}$$

So the difference between standard average of normal players and average containing some null players is defined by a factor dependent only on number of normal players and number of null players:

$$\Delta R(n,d) = \frac{\ln\left(1+\frac{d}{n}\right)}{\ln(\beta)}$$
$$R_t = R_{t0} - \Delta R(n,d)$$

9. Generalization: two teams with different number of players

Considering a game where one player wants to compete with two other in a team is something possible in the Widelands. Also other unequal situations are commonly used by players, since their expected power differs a lot. To allow such games to be valid in rank system another generalization is needed.

First attempt to this problem is to calculate average rating points to each of the teams, regardless of their sizes. If we assume that all the players has the same ranking points R_{Eq}, then after calculating averages, both of them will be the same: $R_1 = R_2 = R_{Eq}$. But smaller team has to performance much better to win the game, so the ranks aren't showing the real power of the team.

To fix that problem we can add some null players introduced in previous point. If the number of null players will be exact to make the teams equally numerous, then the total power of the teams would be comparable. After that both teams should be treated as equally numerous.

In conclusion, to calculate rating points for two teams with different number of players can be easily done. Assuming that teams has n_1 and n_2 players in each team, and $n_2 > n_1$, the ratings should be calculated by equations:

$$R_{t1} = \frac{\ln\left(\sum_{i=1}^{n_1}\beta^{R_i}\right) - \ln(n_1)}{\ln(\beta)} - \frac{\ln\left(1+\frac{n_2-n_1}{n_1}\right)}{\ln(\beta)}$$
$$R_{t2} = \frac{\ln\left(\sum_{i=1}^{n_2}\beta^{R_i}\right) - \ln(n_2)}{\ln(\beta)}$$

Rating deviations should be calculated as standard team work:

$$DR_{t1} = \sqrt{\sum_{i=1}^{n_1}\left(\frac{\beta^{R_i}}{\sum_{j=1}^{n_1}\beta^{R_j}}\right)^2 \cdot RD_i^2}$$
$$DR_{t2} = \sqrt{\sum_{i=1}^{n_2}\left(\frac{\beta^{R_i}}{\sum_{j=1}^{n_2}\beta^{R_j}}\right)^2 \cdot RD_i^2}$$

10. Free-for-all games

Next possible generalization is about games where players don't belong to any team. This situation is called free-for-all and can be also seen as some teams with exactly one player in each of them.

10.1. 1 vs. 1 vs. 1

The easiest case (except direct clash of 2 players) is when we have 3 players on the map fighting for the victory. In this situation on the one side is a player who won the game and two players who lost it. According to point 9., team (1) who won the game and team (2) who lost the game have ratings of:

$$R_{t1} = \frac{\ln(\beta^{R_1}) - \ln(1)}{\ln(\beta)} - \frac{\ln\left(1 + \frac{1}{1}\right)}{\ln(\beta)} = R_1 - \frac{\ln(2)}{\ln(\beta)}$$
$$R_{t2} = \frac{\ln\left(\sum_{i=1}^{2} \beta^{R_i}\right) - \ln(2)}{\ln(\beta)} = \frac{\ln(\beta^{R_2} + \beta^{R_3}) - \ln(2)}{\ln(\beta)}$$

But expected game was not as hard as 1 vs 2 game should be. The team of 2 players was supposed to be not friendly for the whole game and they could fight between each other, helping the winner. So another try is to let player 1 be treated as full strength one:

$$R_{t1} = R_1 = R_1 - \frac{\ln(1)}{\ln(\beta)}$$
$$R_{t2} = \frac{\ln(\beta^{R_2} + \beta^{R_3}) - \ln(2)}{\ln(\beta)}$$

And in this case the rating of winner team is overpowered. The player is supposed to fight with not only one person, but with two of them. Probably not full powered, but it is still more than one front at once.

According to those arguments, none of the solution is acceptable. First one is lowering rank R_1 too much, while second one is not lowering at all. The expected result should be something in the middle. The solution for this situation is in the factor in the equation:

$$R_{t1} = R_1 - \frac{\ln(\alpha)}{\ln(\beta)}$$

The value of α should be between 1 and 2. 1 means that the winning player is fighting alone, while 2 means a player with one null player. So the proposed solution for that is to use fraction of null player, just in the middle of the value, equal to ½ of it (α = 1.5).

$$R_{t1} = R_1 = R_1 - \frac{\ln(1.5)}{\ln(\beta)}$$

10.2. 4 players with no teams

Second situation is about free-for-all game with 4 players. One of them is a winner, 3 of the others are not. Doing the same way of thinking, two border values of α are given:

- α = 3 for a situation when a winner is fighting against all of the others,
- α = 1 for a situation when a winner is fighting against one of them.

To find the correct middle value let's think about standard game of 4 free-for-all. At first players are fighting with each other in teams of two, and one in each team is suppose to lose. Next step is that the winners are supposed to fight with each other and one of them is supposed to win the last fight. In this situation the winner is fighting only with two opponents and α for that should be equal to 2.

10.3. Extrapolating the results

To summarize this point, let's create a table:

Game	Winner rank	α value
1 vs. 1	$R_{t1} = R_1 - \frac{\ln(1)}{\ln(\beta)}$	1
1 vs. 1 vs. 1	$R_{t1} = R_1 - \frac{\ln(1.5)}{\ln(\beta)}$	1.5
4-players	$R_{t1} = R_1 - \frac{\ln(2)}{\ln(\beta)}$	2

The global trend of the α can be described as an equation (dependent on number of all players *n*):

$$\alpha(n) = \frac{1}{2} \cdot n = \frac{1}{2} + \frac{1}{2} \cdot (n-1) = (n-1) - \frac{1}{2} \cdot (n-2)$$

Three equations are shown because they mean something different.

$$\alpha(n) = \frac{1}{2} \cdot n$$

First equation is the simplest version, the easiest to implement in the code.

$$\alpha(n) = \frac{1}{2} + \frac{1}{2} \cdot (n-1) = \frac{1}{2} + \left(1 - \frac{1}{2}\right) \cdot (n-1)$$

Second one is showing that the α value can be interpreted as sum of two values: base one and increasing one. Sum of base and increasing should be equal to 1 (to fulfill situation 1 vs. 1 game), and increasing value is adding to the α value with each of additional opponents (n-1 is equal to winner opponents number).

$$\alpha(n) = (n-1) - \frac{1}{2} \cdot (n-2)$$

Last equation shows another point of view: standard game 1 vs. all others is weaken by a factor ½ multiplied by number of additional players, greater than 1.

10.4. Possible changes to the model

Second and third equations show that the value α can be dependent differently according to different importance on strategy, economy, power and politics in the game (can be different than Widelands). For example if the game is based on politics and real power is a minor thing, the increase value should be much lower, for example ¼:

$$\alpha(n) = \frac{3}{4} + \frac{1}{4} \cdot (n-1) = (n-1) - \frac{3}{4} \cdot (n-2)$$

Also very different equation for the α(n) can be applied, if needed. For example the formula can be a quadratic equation like:

$$\alpha(n) = \frac{1}{4}n^2 - \frac{1}{4}n + 1$$

But then in the point where 3 players lost, the α value equals to 2.5 instead of 2.

11. More than 2 teams with the same number of players in each

Similar situation to the previous point is when more than 2 teams are fighting between each other and all the teams has the same number of players (more than 1).

11.1. 2 vs. 2 vs. 2

The simplest case is when 3 teams of 2 players each are fighting with each other. This is an analogy case to the point 10.1. The same way of thinking can lead to two situations: first when the team should fight with all the others at once, and when the team is fighting with only one team of two. That brings two different equations:

$$R_{t1} = \frac{\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2)}{\ln(\beta)} - \frac{\ln\left(1 + \frac{2}{2}\right)}{\ln(\beta)}$$
$$R_{t1} = \frac{\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2)}{\ln(\beta)} - \frac{\ln\left(1 + \frac{0}{2}\right)}{\ln(\beta)}$$

And according to the previous analysis, the middle value equal to 1 null player is the correct answer:

$$R_{t1} = \frac{\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2)}{\ln(\beta)} - \frac{\ln\left(1 + \frac{1}{2}\right)}{\ln(\beta)}$$
$$\alpha = 1.5$$

11.2. 4 teams of 2

In this situation expected result (and the middle one) is when teams are joined into pairs, one in each pair is defeated and then the winners are fighting with each others. According to that winners were fighting with 4 players (2 teams). The equation for that situation is:

$$R_{t1} = \frac{\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2)}{\ln(\beta)} - \frac{\ln\left(1 + \frac{2}{2}\right)}{\ln(\beta)}$$
$$\alpha = 2$$

11.3. 3 teams of 3

Playing with 9 players in Widelands currently is impossible without any significant changes to the code. But finding out what will happen to the balance in this situation will help to find out general rule for the games with the same number of people in each team (and will be useful in next part of this paper). So two equations for two different looks into the problem shows:

$$R_{t1} = \frac{\ln(\beta^{R_1} + \beta^{R_2} + \beta^{R_3}) - \ln(3)}{\ln(\beta)} - \frac{\ln\left(1 + \frac{3}{3}\right)}{\ln(\beta)}$$
$$R_{t1} = \frac{\ln(\beta^{R_1} + \beta^{R_2} + \beta^{R_3}) - \ln(3)}{\ln(\beta)} - \frac{\ln\left(1 + \frac{0}{3}\right)}{\ln(\beta)}$$

The solution should be as follows:

$$R_{t1} = \frac{\ln(\beta^{R_1} + \beta^{R_2} + \beta^{R_3}) - \ln(3)}{\ln(\beta)} - \frac{\ln\left(1 + \frac{1.5}{3}\right)}{\ln(\beta)}$$
$$\alpha = 1.5$$

11.4. General rule

Let's join all the results into one table:

teams \ players	1	2	3
2	1	1	1
3	1,5	1,5	1,5
4	2	2	

First column is labeling how many teams are fighting, while first row is showing how many players are in each team. If there are 2 teams only, modifier α should be equal to 1 (ln(1) = 0).

From the collected data it can be seen, that the value of α is not dependent on number of players in the teams, even the origins of the values are a bit different (examples for 3rd and 4th rows):

$$\alpha_3 = 1 + \frac{0.5}{1} = 1 + \frac{1}{2} = 1 + \frac{1.5}{3}$$
$$\alpha_4 = 1 + \frac{1}{1} = 1 + \frac{2}{2}$$

In conclusion, if there are M teams with N players in each of them, the equation for winner team average points is:

$$R_{t1}(M,N) = \frac{\ln\left(\sum_{i=1}^{N} \beta^{R_i}\right) - \ln(N)}{\ln(\beta)} - \frac{\ln\left(\frac{M}{2}\right)}{\ln(\beta)}$$

12. Game with different number of players in more than 2 teams.

The most general case is when players are split into any number of teams (larger than 1) and the teams can contain any number of players (at least 1 in each).

Some of the cases can be covered by the rules described above, for example 6 players playing in 3 , 2 players in each. Or (probably most common situation) 2 players fighting with each other.

But not all of the cases are like that. For example we can think of a game where 6 players are split into 3 teams: 1 player playing alone, 2 players in second team and 3 players in the last one. To solve this situation several steps are needed to apply:

- Add some null-players to make the teams comparable
- Join all players into two teams: winners and losers
- Calculate summarize team averages with additional null player for winner team (if needed)

12.1. Example of use

Example shown above (teams: 1, 2 & 3 players in each), and let's add an information that team of 2 have won the match.

First step is to add null-players to each of the teams to make them comparable:
- Team of 1 players (R_1) now is a team of 3 players (R_1, 0, 0),
- Team of 2 players (R_2, R_3) now is a team of 3 players (R_2, R_3, 0),
- Team of 3 players (R_4, R_5, R_6) stays the same.

Second step is to join players into two teams:
- Winners: (R_2, R_3, 0),
- Losers: (R_1, R_4, R_5, R_6, 0, 0).

Last step is to calculate team averages, also with additional α term:
Since there are 3 teams, α = 1.5. Also null players has $\beta^R = 0$, so that terms can be ommited.

$$R_{t1} = \frac{\ln(\beta^{R_2} + \beta^{R_3}) - \ln(3)}{\ln(\beta)} - \frac{\ln(1.5)}{\ln(\beta)}$$

$$R_{t2} = \frac{\ln(\beta^{R_1} + \beta^{R_4} + \beta^{R_5} + \beta^{R_6}) - \ln(6)}{\ln(\beta)}$$

13. Applying results for more players at once

After ending a game new stats should be applied to a player. Comparing to the old ratings and the new ones, the difference between stats can be calculated. Those values are useful also for more complex games than 1 vs. 1.

At first it is needed to change the complex, multi team situation, into 1 vs. 1 game with comparable rank points. Then for all teams (winners & losers) new collected ranks should be calculated (according to standard Elo / Glicko rules). With new and old rank points differences can be calculated too. The last thing is applying differences in ranks to all players (with proper value differences). After that procedure new collected ranks will be exactly changed by differences calculated before.

This is reasoned by point 4.4. which tells that if all players in a team gain (or lose) the same amount of points, the collected rank will also change by the same amount of points.

14. References

[1] https://en.wikipedia.org/wiki/Elo_rating_system
[2] https://en.wikipedia.org/wiki/Glicko_rating_system
[3] http://www.ets.org/Media/Research/pdf/RM-15-03.pdf
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Any questions? Just ask.
Any mistakes? Please point them out!

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