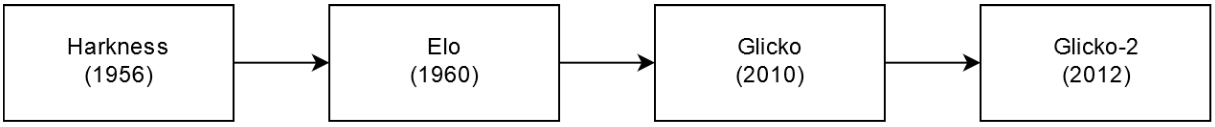
1. Introduction

Rating systems along the history were different, but all of them were trying to compare two unknown people in one field. Probably the most famous attempt to that problem was Elo rating system[1], widely used in chess games. But Elo rating system was improved and introduced as Glicko rating system[2]. The full improvements history can be shown as:



2. Power of a player

It is known that two players in the game can differ a lot with their power in the competition. For them it can be even several orders of magnitude of power difference (like 1 over 1000). But the pointing systems above are handling with that. How? As we can read in Wikipedia[1]:

$$Q = 10^{\frac{R}{400}}$$

Where Q - power of a player, R - his/her rating points

That means if there are two players with a difference of 400 points, the stronger one should be exactly 10 times stronger than the weaker one.
And since 10^(1/400) ≈ 1.005773063, the equation can be simplified to:

$$Q = 1.005773063^{R}$$
$$\beta = 1.005773063$$
$$Q = \beta^{R}$$

3. Two player teams

Since Widelands is not simple 1 vs 1 game only, we need to calculate the correct average of the players points. And since the power of the player is not linear, but exponential, the average power of two players is not a simple arithmetic mean.

To get proper equation, we need to determine our goal. For two players with the points R_1 and R_2, we need to find two exactly equal opponents with rating R_3 that their total power is equal to total power of R_1 and R_2:

$$Q_1 + Q_2 = Q_3 + Q_3 = 2 \cdot Q_3$$

Using equation above we get:

$$Q_1 + Q_2 = \beta^{R_1} + \beta^{R_2}$$
$$2 \cdot Q_3 = 2 \cdot \beta^{R_3}$$
$$\beta^{R_1} + \beta^{R_2} = 2 \cdot \beta^{R_3}$$
$$\frac{1}{2}(\beta^{R_1} + \beta^{R_2}) = \beta^{R_3}$$

Solving the equation is by using natural logarithm[3] ln on both sides:

$$\ln\left(\frac{1}{2}(\beta^{R_1} + \beta^{R_2})\right) = \ln(\beta^{R_3})$$
$$\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2) = R_3 \cdot \ln(\beta)$$

$$R_3 = \frac{\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2)}{\ln(\beta)}$$

It can be also written as (assuming $R_1 < R_2$):

$$R_3 = R_1 + \frac{\ln(1 + \beta^{R_2 - R_1})}{\ln(\beta)} - \frac{\ln(2)}{\ln(\beta)}$$

4. Examples of usage

a) Same power

Let's assume that $R_1 = R_2$, then:

$$R_3 = \frac{\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2)}{\ln(\beta)} = \frac{\ln(\beta^{R_1} + \beta^{R_1}) - \ln(2)}{\ln(\beta)} = \frac{\ln(2 \cdot \beta^{R_1}) - \ln(2)}{\ln(\beta)} =$$
$$= \frac{\ln(2) + \ln(\beta^{R_1}) - \ln(2)}{\ln(\beta)} = \frac{\ln(\beta^{R_1})}{\ln(\beta)} = \log_{\beta}(\beta^{R_1}) = R_1$$

So as we expected, the formula gets us to the same rating points as the initial players.

b) Difference of 200 (decent difference, but not huge one)

Let's assume that $R_2 = R_1+200$, then:

$$R_3 = R_1 + \frac{\ln(1 + \beta^{R_2 - R_1})}{\ln(\beta)} - \frac{\ln(2)}{\ln(\beta)} = R_1 + \frac{\ln(1 + \beta^{R_1 + 200 - R_1})}{\ln(\beta)} - \frac{\ln(2)}{\ln(\beta)} =$$
$$= R_1 + \frac{\ln(1 + \beta^{200})}{\ln(\beta)} - \frac{\ln(2)}{\ln(\beta)} = R_1 + \frac{\ln(1 + 3.162278 \ldots)}{\ln(1.00577 \ldots)} - \frac{\ln(2)}{\ln(1.00577 \ldots)} =$$
$$= R_1 + 247.732 \ldots - 120.412 \ldots \approx R_1 + 127.3$$

So the result is that the expected power of two players is about 127 points higher than the weaker player or about 73 points lower than the stronger one.

c) Finding difference that doesn't affect the stronger player

Last part is about finding the rating that is not affecting the result rating of stronger player and the weaker one. In mathematical language can be written as:

$$R_1 + R_2 \approx 0 + R_2$$

To finish this equation we need to define approximately sign. From a rating system, two players are equal power when their amount of points differs 1 at most. Applying that:

$$2 \cdot R_3 = R_2 + 0$$
$$2 \cdot (R_3 + 1) = R_2 + R_1$$

So let's find out what R_3 equals to:

$$R_3 = \frac{\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2)}{\ln(\beta)} = \frac{\ln(\beta^{0} + \beta^{R_2}) - \ln(2)}{\ln(\beta)} = \frac{\ln(1 + \beta^{R_2}) - \ln(2)}{\ln(\beta)}$$

Since R_2 is expected to be around 1000 or more, the value of β^{R_2} is expected to be much greater than 1:

$$\beta^{R_2} \gg 1$$

So applying that, we can calculate:

$$R_3 = \frac{\ln(1 + \beta^{R_2}) - \ln(2)}{\ln(\beta)} \approx \frac{\ln(\beta^{R_2}) - \ln(2)}{\ln(\beta)} = R_2 - 120.412 \ldots$$

The result is that half of power of sufficient rating score is equal to score decreased by about 120 points. Going further, we apply those numbers to the second equation:

$$R_3 + 1 = R_2 - 120.412 + 1 = R_2 - 119.412 = R_2 - \gamma$$

$$\gamma = 119.412\ldots$$

$$R_2 + R_1 = \frac{\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2)}{\ln(\beta)} = R_2 - \gamma$$

$$\ln(\beta^{R_1} + \beta^{R_2}) - \ln(2) = R_2 \cdot \ln(\beta) - \gamma \cdot \ln(\beta)$$

$$\ln(\beta^{R_1} + \beta^{R_2}) = R_2 \cdot \ln(\beta) - \gamma \cdot \ln(\beta) + \ln(2)$$

$$\beta^{R_1} + \beta^{R_2} = \exp(R_2 \cdot \ln(\beta) - \gamma \cdot \ln(\beta) + \ln(2))$$

$$\beta^{R_1} = \exp(R_2 \cdot \ln(\beta) - \gamma \cdot \ln(\beta) + \ln(2)) - \beta^{R_2}$$

$$R_1 = \frac{\ln(\exp(R_2 \cdot \ln(\beta) - \gamma \cdot \ln(\beta) + \ln(2)) - \beta^{R_2})}{\ln(\beta)}$$

$$= \frac{\ln\left(\frac{\exp(R_2 \cdot \ln(\beta))}{\exp(\gamma \cdot \ln(\beta))} \cdot \exp(\ln(2)) - \beta^{R_2}\right)}{\ln(\beta)} =$$

$$= \frac{\ln\left(\frac{\beta^{R_2}}{\beta^{\gamma}} \cdot 2 - \beta^{R_2}\right)}{\ln(\beta)} = \frac{\ln\left(\beta^{R_2}\left(\frac{2}{\beta^{\gamma}} - 1\right)\right)}{\ln(\beta)} = \frac{\ln(\beta^{R_2}) + \ln\left(\frac{2}{\beta^{\gamma}} - 1\right)}{\ln(\beta)} =$$

$$= \frac{\ln(\beta^{R_2})}{\ln(\beta)} + \frac{\ln\left(\frac{2}{\beta^{\gamma}} - 1\right)}{\ln(\beta)} = \frac{\mathrm{R}_2 \cdot \ln(\beta)}{\ln(\beta)} + \frac{\ln\left(\frac{2}{\beta^{\gamma}} - 1\right)}{\ln(\beta)} = R_2 + \frac{\ln\left(\frac{2}{\beta^{\gamma}} - 1\right)}{\ln(\beta)}$$

$$R_1 \approx R_3 - 895.44$$

So if the weaker player has lower rating by about 900 points, this doesn't matter to global rating for both players.

5. References

[1] https://en.wikipedia.org/wiki/Elo_rating_system

[2] https://en.wikipedia.org/wiki/Glicko_rating_system

[3] https://en.wikipedia.org/wiki/Natural_logarithm

[4] https://www.widelands.org/forum/topic/4583/?page=3#post-28850

[5]